Reverse-flow integral methods for second-order supersonic flow theory

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(Received 15 July 1963)

A general reverse-flow relation is obtained within the framework of second-order (in surface deflexion) supersonic flow theory. From this it is shown that the *second*-order increment in the drag of an arbitrary quasi-cylindrical body can be expressed as surface and volume integrals of the *first*-order solutions corresponding to forward and reverse flow past the body. Analogous results are obtained for second-order transverse forces and moments on an arbitrary quasi-planar wing, where the first-order reverse flow must correspond to certain zero-thickness wings. Other similar results are possible. Thus, second-order aerodynamic forces on bodies may be obtained from first-order solutions by quadrature. It is also shown that the reverse-flow integral relation can yield the pressure distribution on the surface by inversion of an integral equation constructed therefrom. It is thought that these results should be particularly useful for the Mach-number range between that of linearized theory and that of full hypersonic small-disturbance theory.

1. Introduction

A general reverse-flow relation is a completely formal integral connexion between the flow past a body of interest (designated the forward flow) and the reverse flow past a body that may be the same or different. By a suitable choice of the reverse-flow body, one can contrive, from the general relation, useful expressions or results concerning the forward flow. Reverse-flow relations are a well-known element in the classical linearized theory of compressible flow (see Ward 1955 for quasi-cylinders and Clarke 1959 for general bodies). In this paper we present a general reverse-flow relation, and a number of its consequences, within the framework of second-order (in surface deflexion) supersonic flow theory. Parts of the formalism and technique are analogous to that employed in the first-order theory. However, the nature of the relation and results obtained are distinct—they do not have a precedent in first-order theory.

The non-linear potential equation for compressible, perfect-gas flow is well known. Second-order theory comprises the second step in an iteration or expansion procedure in terms of the surface deflexion parameter; the first step gives linearized theory. Since the specious singularities of linearized theory are usually worsened by the ordinary procedure and require rather elaborate adjustment, the second step is a big step in terms of labour as well as accuracy. Adjusted or uniform or uniformly valid second-order theory accounts for much thicker bodies at moderate supersonic Mach number than linearized theory can account for. It is perhaps more important that, for a given thin body, it also accounts for the Mach number range between that of linearized theory and that of full hypersonic small-disturbance theory (see Clarke 1962); in practice, this intermediate Mach-number range is perhaps 2·5 to 6.

The linearized potential must always be continuous but the first derivatives can be infinite at stagnation points, sharp wing edges, and the edges of vortex sheets; the second derivatives can be infinite (admitting the smoothed sense) on waves. A number of schemes for obtaining a spatially uniform second approximation is particular cases has been offered, but the smoothing and analytic-continuation arguments have limited applicability; the principal techniques are associated with the work of Lighthill. Denoting by τ the deflexion parameter for quasi-cyclinders (say), he proposes the expansion of the exact perturbation potential Φ according to $\Phi(x,y,z) = \tau\Phi_1(u,y,z)+\tau^2\Phi_2(u,y,z)+\ldots$, where the new co-ordinate u is related implicitly to the Cartesian co-ordinates (x,y,z) by $x = u+\tau x_1(u,y,z)+\tau^2 x_2(u,y,z)+\ldots$. The functions x_i are to be determined so as to obviate the difficulties. It is sometimes convenient to interpret (u,y,z) as locally distortable curvilinear co-ordinates in the (x,y,z)-space, and sometimes as a new rectangular space. Within this framework, a number of 'shift rules' have been developed, according to which the ordinary first- and second-order solutions may be correctly adjusted and shocks then inserted between any limit surfaces. The reader may refer to discussions and/or extensions of these procedures by Lighthill (1954), Wallace & Clarke (1963), and Clarke & Wallace (1963). In the second reference it is further suggested that it is often adequate to make the second-order solution uniformly valid only to first order (x_1 non-zero and x_2 disregarded). At any rate, the need for adjustment is always signalled by singularities in the ordinary first- or second-order solution. The last named can be not only wrong but internally inconsistent—in the sense that its irrotational equations can give vortex sheets (potential jumps) in the fluid interior where shocks belong nearby.

In accord with these remarks, we shall obtain the general reverse-flow relation, provide interpretations, and deduce results within the framework of ordinary first- and second-order theory. The defects are then to be adjusted in the course of the development where called for by the singularities. The alternative procedure would be to use the (u,y,z) co-ordinates and Lighthill's formalism. Such results would not be edifying. We shall not attempt in this paper an extensive résumé of the theory touched on in the preceding paragraphs. The reader may consult the references cited for details; the terminology employed here is consistent with that used by Clarke (1962), who has also listed the forms of the various equations needed for each class of body. However, the background material is not really needed to follow the development in its essentials.

From the general reverse-flow relation, we shall show that the *second*-order increment in the drag of an arbitrary quasi-cylindrical body can always be expressed as surface and volume integrals of the *first*-order solutions corresponding to forward and reverse flow past the body. We next obtain analogous results for the second-order transverse forces and moments on an arbitrary quasi-planar wing;

the first-order reverse flow in this case must correspond to certain zero-thickness wings. Thus, without solution of the second-order problem, the second-order aerodynamic forces on quasi-cylinders may be obtained from first-order solutions by quadrature; the only co-ordinates appearing in the integrals are the co-ordinates of integration. It is expected that these integrals would generally be carried out numerically in particular cases. Corresponding results for fusiform and other bodies could be written, but not with the same generality because the second-order circumferential flow does not enter the reverse-flow relation in the necessary manner; these bodies require further study. It is then pointed out that the results discussed above do not have to be adjusted when the singularities are due to wave phenomena. We also show in the paper how the reverse-flow integral can be made to yield the forward-flow pressure distribution on the surface. By suitable choice of the reverse flow, one can obtain an integral equation in this pressure wherein the kernel and the integrals to be evaluated are controllable by the aforesaid choice. In a companion paper (Clarke & Wallace 1963), this technique is applied to the delta wing with supersonic edges and a fully analytic, uniform second-order solution for surface pressure is obtained.

The development of second-order theory has been impeded because of the difficulty in computing the effect of the spatial source distribution, whose strength is determined by the inhomogeneous terms in the differential equation. One would need particular integrals more general than those discovered by Van Dyke (1952) to remove these terms in the equation. The above integral-equation formulation, made possible by the reverse-flow method, offers a new method of attack.

First-order solutions are always presumed known herein. Similarly, it is assumed throughout that the first-order aerodynamic forces have been or will be obtained either by direct integration or from the usual reverse-flow relations of linearized theory; these often reduce or eliminate the requisite labour. It is possible to simplify the volume integrals that occur in the paper with certain devices developed in the companion paper. We shall not utilize them here because, however useful, they tend to get in the way of the essential points.

2. General second-order reverse-flow relation

We consider the supersonic steady flow of a perfect gas past a body (or bodies) fixed in a Cartesian frame (x, y, z). In the undisturbed region upstream, the flow is uniform with pressure p_0, density ρ_0, Mach number $M > 1$, and velocity $\mathbf{U} = U\mathbf{i}$, where $U \gtrless 0$ and $\mathbf{i}$ is a unit vector in the x-direction. Similarly, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the y- and z-directions, respectively. The equation of the body surface contains a small affine parameter τ, on which the solution therefore depends. If, as $\tau \to 0$, the surface collapses on to a cylinder with streamwise generators, then the body is called a quasi-cylinder. If it collapses to a streamwise line, the body is called a fusiform body. The latter's surface may contain anomalies like shoulders, near which the flow is quasi-cylindrical in nature. But when this surface is sufficiently smooth the body is then called a slender body. When both surfaces and lines appear in the limit, the body is a general body. With $\mathbf{q}$ the local velocity, we write

$$\mathbf{q} = \mathbf{U} + \mathbf{V}, \tag{1}$$

where $\mathbf{V}$ is the full perturbation velocity to second order in τ. To this order $\mathbf{V}$ will always be irrotational. A number of different series representations of $\mathbf{V}$ are generally required to express $\mathbf{V}$ throughout the field; these can be expected to overlap.

The equations governing $\mathbf{V} = \nabla\phi$ may be written

$$\nabla . \mathbf{W} = Q \quad \text{and} \quad \nabla \times \mathbf{V} = 0, \tag{2a, b}$$

where

$$\mathbf{W} \equiv \mathbf{\Psi} . \mathbf{V}, \quad \mathbf{\Psi} \equiv \begin{bmatrix} -B^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

$B = \sqrt{(M^2 - 1)}$, and Q is determined by the first-order result in accord with the literature. These equations amount to the inhomogeneous wave equation in ϕ; they may be manipulated with the convenience of the equations of potential theory. On the body with outward normal $\mathbf{n}$, the boundary condition is the appropriate approximation of $(\mathbf{U} + \mathbf{V}) . \mathbf{n} = 0$. The shock relations require that the jump in velocity at shock waves with normal $\mathbf{n}$ be in accord with

$$[\mathbf{n} \times \mathbf{V}] = 0 \quad \text{or} \quad [\phi] = 0 \tag{4}$$

and an approximate version of

$$[\mathbf{n} . \mathbf{q}] = \frac{2}{\gamma + 1}\left(\frac{a^2}{\mathbf{n} . \mathbf{q}} - \mathbf{n} . \mathbf{q}\right), \tag{5}$$

where the symbol [ ] denotes the indicated jump on passing through the wave, a is the speed of sound, γ is the ratio of specific heats, and the right member of (5) is to be evaluated on the upstream side. The ordinary second-order solutions do not contain the freedom to impose (4) and (5), and show non-uniformities near the neighbouring free-stream characteristics, these being the characteristics of the ordinary equations. The initial condition is that $\mathbf{V}$ must vanish everywhere upstream of the foremost envelope of disturbance. The written-out versions of all the relations just discussed treat U algebraically and hold for both directions of flow, except for the interpretation of 'upstream' in the initial condition. They also describe the first-order flow if the higher-order terms are discarded.

Consider the forward flow $U = U_F > 0$ past a body (or bodies) producing a perturbation velocity $\mathbf{V}_F$. Figure 1 gives some generalized sketches which include types of surfaces that might be present in a particular problem. Also consider a reverse flow $U = U_R = -U_F$ past a body which is the same as or different from the body in forward flow and at our disposal. The forward and reverse flows are denoted by subscripts F and R, respectively, and have the same free stream Mach number and density. A useful connexion between the two flows is provided by the scalar product of $\mathbf{U}_F$ and the volume-surface integral identity (Ward 1955, p. 222)

$$\oint_{A'} (\mathbf{V}_F \mathbf{W}_R . \mathbf{n} + \mathbf{V}_R \mathbf{W}_F . \mathbf{n} - \mathbf{V}_F . \mathbf{W}_R \mathbf{n})\, ds$$
$$= -\int_{T'} \{\mathbf{V}_F \nabla . \mathbf{W}_R + \mathbf{V}_R \nabla . \mathbf{W}_F - \mathbf{W}_F \times (\nabla \times \mathbf{V}_R) - \mathbf{W}_R \times (\nabla \times \mathbf{V}_F)\}\, dT, \tag{6}$$

where the region T' is bounded by the closed surface A' with unit inward normal $\mathbf{n}$. This identity is a consequence of the divergence theorem that contains the operators of equations (2); the vectors appearing must be continuous and differentiable once over T'.

Choose two finite regions (collectively called T), one above the body and one below. Let the upper region be bounded by (i) the interior sides of the envelopes of disturbance in forward and reverse flow (see figure 1), (ii) the upper true surface of the body in forward flow (depending on technique, it might be useful to enclose any edge singularities within tubes of vanishing radius), (iii) the upper surface of any vortex sheets present, and (iv) a closing surface lying near the (x, y)-plane. The boundaries of the lower region are similar and the lower closing surface is contiguous with the upper. Now add the result for each region. The

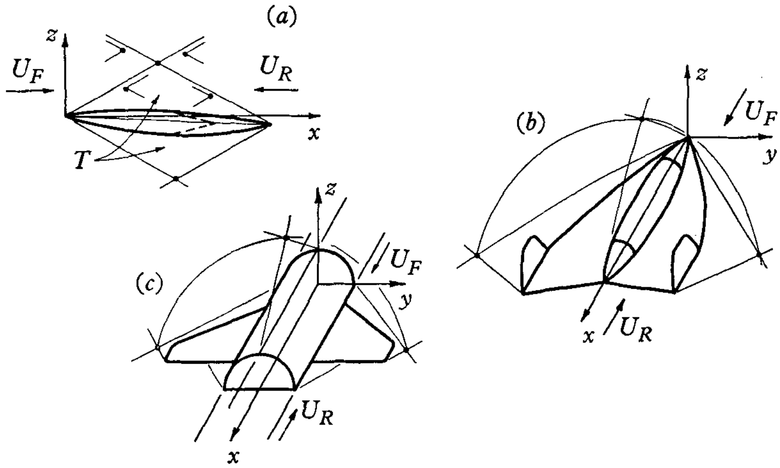

FIGURE 1. Bodies and envelopes of disturbance due to forward and reverse flow.

integral over the upper closing surface is cancelled by its lower counterpart since the vectors are continuous and the normals are opposite. Let $\mathbf{V}_F$ satisfy (2) and $\mathbf{V}_R$ satisfy (2) with $Q \equiv 0$. Then the reverse flow satisfies the *first*-order equations and the integral over T contains only first-order quantities. The envelopes of disturbance are in general surfaces of discontinuity in velocity. *If* these discontinuities are such that the surface integrand in (6) is *continuous*, then the integrals over the envelopes of disturbance in forward and reverse flow will vanish because on their exterior sides either the forward or reverse perturbation is zero. With the same supposition, velocity discontinuities across waves in the interior of T do not contribute a surface integral to the result.

The conormal $\boldsymbol{\nu}$ is related to the unit surface normal $\mathbf{n}$ by the definition $\boldsymbol{\nu} = \boldsymbol{\Psi} . \mathbf{n}$. On an undisturbed characteristic, $\boldsymbol{\nu} . \mathbf{n} = 0$ and $\boldsymbol{\nu}$ lies along the wave such that its projection on the (y, z)-plane coincides with the projection of $\mathbf{n}$. Within the framework of a distinctly different argument, Ward (1955, pp. 13, 27, 72, 87 and 222) proves that sufficient conditions for the surface integrand in (6) to be continuous across a surface supporting a velocity discontinuity in one of the flow directions are that

$$[\mathbf{n} \times \mathbf{V}] = 0 \quad \text{and} \quad [\mathbf{n} . \mathbf{W}] = 0. \qquad (7a, 8a)$$

Now these may be respectively written

$$[\phi] = 0, \quad \text{and} \quad [\boldsymbol{\nu}.\mathbf{V}] = 0 \quad \text{or} \quad \partial[\phi]/\partial\nu = 0. \qquad (7b, 8b)$$

Thus (7) is sufficient for (8) when the surface is an undisturbed characteristic, while conditions (7) and (4) coincide. A first-order field, and specifically the reverse flow, will always fulfill ($7b$) and hence (7) and (8). But ordinary second-order solutions sometimes fail to yield ($7b$). In second-order theory, one distinguishes between waves consisting of an undisturbed Mach cone (e.g. cone flow) and waves consisting of an envelope of undisturbed Mach cones (e.g. wedge flow). We may place a thin disk or pillbox along a wave and integrate ($2a$) over the disk volume to verify ($8a$) from the behaviour of Q. The first sort of wave will satisfy ($8a$) while the second generally will not; and (8) is necessary for (7). The result $[\phi] \neq 0$ is never admissible. It is sometimes possible to smooth the body and approach the actual shape by a limiting process, but this is not generally satisfactory. We are then to render the field uniform and, in so doing, we fulfill ($7b$) across the correct discontinuity. This will still lie along the undisturbed Mach cone in the rectangular (u, y, z)-space; $\boldsymbol{\nu}$ will still lie along the wave so (8) will be fulfilled when (7) is. We therefore conclude that the surface integrand in (6) is always continuous across shock waves of the uniform second-order field.

The general second-order reverse-flow relation then is

$$\int_{A_b+A_v} (\mathbf{U}_F.\mathbf{V}_F\mathfrak{W}_R.\mathbf{n} - \mathbf{U}_R.\mathfrak{V}_R\mathbf{W}_F.\mathbf{n} - \mathbf{V}_F.\mathfrak{W}_R\mathbf{U}_F.\mathbf{n})\,ds = \int_T \mathbf{U}_R.\mathfrak{V}_R Q\,dT, \qquad (9)$$

where A_b is the surface of the body (or bodies) in forward flow, A_v is the surface of any vortex sheets appearing within the envelopes of disturbance in forward and reverse flow, and $\mathbf{n}$ is the outward normal to A_b or A_v. We have also introduced the notation where a first-order velocity perturbation is denoted by $\mathfrak{V} \equiv \nabla\varphi$ instead of $\mathbf{V} \equiv \nabla\phi$, and where $\mathfrak{W} \equiv \Psi.\mathfrak{V}$.

Equation (9) contains the two successive orders of magnitude. For quasi-cylinders the first two integrals are $O(\tau^2)$ and the second two are $O(\tau^3)$; for slender bodies the first three integrals are $O(\tau^4)$ and the fourth is $O(\tau^6)$. The choice of a reverse flow that satisfies $\nabla.\mathfrak{W} = 0$ and $\nabla \times \mathfrak{V} = 0$ is essential because it suppresses the forward flow over T and thereby makes possible useful deductions from (9). The choice also gives us a more manageable reverse flow that we can use in the intended manner (see § 1). Because of the formal nature of (6), we are still free to choose the boundary conditions the reverse flow is to satisfy. In first-order theory, there are usually several variants that are equivalent within the allowable error: on a quasi-cylinder, for example, one can write $\mathfrak{V}.\mathbf{n} = -\mathbf{U}.\mathbf{n}$, $\mathfrak{W}.\mathbf{n} = -\mathbf{U}.\mathbf{n}$, $\mathfrak{V}.\boldsymbol{\nu} = -\mathbf{U}.\mathbf{n}$, or $\mathfrak{V}.\mathbf{N} = -\mathbf{U}.\mathbf{n}$ (where $\boldsymbol{\nu}$ is the normal to the cylindrical reference surface and $\mathbf{N}$ is the normal to the cross-section perpendicular to $\mathbf{U}$). We identify the parameter τ in the reverse flow with the small parameter τ in the forward flow, because this is convenient. Then any subsequent approximations made in the first-order reverse-flow problem, as defined by the equations and chosen boundary conditions, must be in accord with the allowable error in the second-order forward-flow problem; it is seen that a second-order modification in the reverse flow will count in (9) in the larger terms. This error feature is clear in the suitably defined first-order forward-flow problem: if we consider the formulation which seeks a second-order *incremental* improvement to it, then we know a second-order error introduced in it will negate the improvement sought. We will use the reverse-flow boundary conditions on body and vortex sheets that are consistent with the first- and second-order forward-flow problem. The first-order version of (9) is obtained by setting $Q = 0$, and replacing $\mathbf{V}_F$ by $\mathfrak{V}_F$ and $\mathbf{W}_F$ by $\mathfrak{W}_F$. The incremental second-order formulation then follows by subtraction.

If the body in reverse flow is not the same as the one in forward flow, it might be as shown dashed in figure 1 (*a*). For the bodies sketched in figure 1 (*b*) and 1 (*c*), a wing alone and without thickness might be convenient, as it was for the analysis of the linearized counterpart of 1 (*c*) carried out by Clarke (1960). The interpretation of (9) has to be given separately according to whether the flow is to be analysed on the true surface of the body or on a cylindrical reference surface nearby. This is done in the next two sections. The general body defined represents the combined case.

3. Interpretation for fusiform bodies

The terms $\mathfrak{W}_R . \mathbf{n}$ and $\mathbf{W}_F . \mathbf{n}$ on A_b in (9) resemble slopes and $\mathbf{U}_F . \mathbf{V}_F$ is a fragment of the second-order pressure relation. Equation (9) is useful for fusiform bodies when (i) the second, third, and fourth integrals can all be evaluated in terms of the first-order forward and reverse flow (this is already the case for the fourth) and (ii) the term $\mathbf{U}_F . \mathbf{V}_F$ is the essential part of the pressure relation. In this case (9) evaluates a weighted integral of $\mathbf{U}_F . \mathbf{V}_F$.

We discuss boundary conditions and surface geometry in the plane of $\mathbf{n}$ and $\mathbf{U}$ at a point P on the body surface, as shown in figure 2 (*a*). The unit vector $\mathbf{N}$ is normal to the cross-section through P perpendicular to $\mathbf{U}$, $\mathbf{C} \equiv \mathbf{i} \times \mathbf{N}$, and $\cos\delta \equiv \mathbf{n} . \mathbf{N}$. Because $\mathbf{n} . \mathbf{C} = 0$ we obtain from $\mathbf{q} . \mathbf{n} = 0$ the tangency condition in second-order theory

$$\mathbf{V} . \mathbf{N} = \tan\delta \quad (U + \mathfrak{V} . \mathbf{i}), \tag{10}$$

and first-order theory

$$\mathfrak{V} . \mathbf{N} = U \tan\delta, \tag{11}$$

for either flow direction. The vectors in (10) and (11) are to be evaluated on the true surface. From (3) we obtain the exact connexion

$$\mathbf{W} . \mathbf{n} \equiv \mathbf{W} . \mathbf{i} \mathbf{n} . \mathbf{i} + \mathbf{W} . \mathbf{N} \mathbf{n} . \mathbf{N} \equiv B^2 \sin\delta \, \mathbf{V} . \mathbf{i} + \cos\delta \, \mathbf{V} . \mathbf{N}. \tag{12}$$

The vectors $\mathfrak{W}$ and $\mathfrak{V}$ have the same connexion. From (10) and (12) we evaluate the cited second-order forward-flow term in (9) as

$$\mathbf{W}_F . \mathbf{n} = -\mathbf{n} . \mathbf{i} [U_F + (B^2 + 1) \mathfrak{V}_F . \mathbf{i}]. \tag{13}$$

Similarly, we re-express, to the correct approximation, the cited first-order term for reverse flow past the same surface, say, as

$$\mathfrak{W}_R . \mathbf{n} = -\mathbf{n} . \mathbf{i} [U_R + B^2 \mathfrak{V}_R . \mathbf{i}]. \tag{14}$$

It is useful to note that the first product in (9) may now be simplified to

$$\mathbf{U}_F . \mathbf{V}_F \mathfrak{W}_R . \mathbf{n} = -U_R \mathbf{U}_F . \mathbf{V}_F \mathbf{n} . \mathbf{i} - B^2 \mathbf{U}_F . \mathfrak{V}_F \mathbf{n} . \mathbf{i} \mathfrak{V}_R . \mathbf{i}, \tag{15}$$

so that only the first term in (15) contains the second-order flow. The same sort of simplification in accord with the allowable error gives for the first product in the third integral of (9)

$$\mathbf{V}_F.\mathfrak{W}_R = \mathbf{V}_F.\mathbf{N}\mathfrak{B}_R.\mathbf{N}+\mathbf{V}_F.\mathbf{C}\mathfrak{B}_R.\mathbf{C}-B^2\mathfrak{B}_F.\mathbf{i}\mathfrak{B}_R.\mathbf{i}. \tag{16}$$

For fusiform (and slender) bodies, the pressure relation for the second-order forward flow is

$$(p_0-p_F)/\rho_0 = \mathbf{U}_F.\mathbf{V}_F+\tfrac{1}{2}(\mathbf{V}_F.\mathbf{N})^2+\tfrac{1}{2}(\mathbf{V}_F.\mathbf{C})^2+\text{first-order terms}. \tag{17}$$

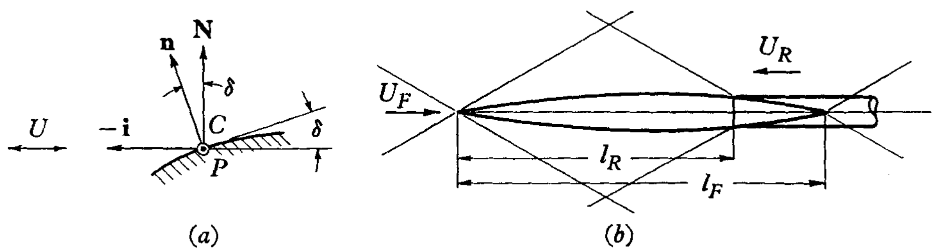

FIGURE 2. (*a*) Plane of **n** and **U** at point P on body surface. (*b*) Forward and reverse fusiform bodies with partly common surfaces.

To see the significance of these formalities, consider an inclined body of revolution for which $\mathbf{V}_F.\mathbf{C} = 0$. Since $\mathbf{V}_F.\mathbf{N}$ is always known from (10), then (16) is known, and (9) evaluates for us the integral

$$\int_{A_b} \mathbf{U}_F.\mathbf{V}_F\mathbf{n}.\mathbf{i}\,ds \tag{18}$$

if the reverse body is identical. Now the direct calculation of the second-order drag

$$D_F = -\int_{A_b}(p_F-p_0)\,\mathbf{n}.\mathbf{i}\,ds \tag{19}$$

using (17) requires direct calculation of only (18) from the second-order solution, since the other terms depend on the first-order forward flow; but we have just seen that (9) evaluates (18) in terms of first-order results! Therefore (9) renders the second-order solution unnecessary. If we modify the reverse body as shown in figure 2(*b*), then the evaluation of (18) given by (9) contains the generic distance to the shoulder of the cylindrical forebody in the reverse flow. (In this case, A_b is to be interpreted as the surface of the forward body up to the shoulder.) We, in fact, get a well-posed integral equation in the current variable l_R; the inversion gives $\mathbf{U}_F.\mathbf{V}_F$ on A_b, the essential part of the pressure distribution. When our forward body is inclined, these connexions are spoiled because of the occurrence of $\mathbf{V}_F.\mathbf{C}$ in (16) and (17), although small cross-flows in the sense of Van Dyke (1951) can be considered. For other fusiform bodies these arguments must be buttressed by some knowledge of $\mathbf{V}_F.\mathbf{C}$ on A_b. We shall not consider fusiform bodies further here, because the equivalent interpretations of (9) carry through quite generally and more simply for quasi-cylinders.

4. Interpretation for quasi-cylinders

The flow on the surface of a quasi-cylinder can be related by Maclaurin-series expansion to the flow on the cylindrical reference surface that lies near the true surface for small τ. To second-order, $O(\tau^2) \ll 1$. The lateral dimension or dimensions of the cylinder or skeleton are not small. Examples of quasi-cylinders are certain wings, ducts and wing-body combinations (but the body cannot close upstream or downstream). With the problem transferred to the reference surface, we may reinterpret the argument leading to (9) such that this surface is meant instead of A; T is then somewhat different. Thus, from (9), the required version of the second-order reverse-flow relation is

$$\int_{\Sigma_b+\Sigma_v} (\mathbf{U}_F \cdot \mathbf{V}_F \mathfrak{B}_R \cdot \boldsymbol{\nu} - \mathbf{U}_R \cdot \mathfrak{B}_R \mathbf{V}_F \cdot \boldsymbol{\nu})\, ds = \int_T \mathbf{U}_R \cdot \mathfrak{B}_R Q \, dT, \tag{20}$$

where Σ_b is the reference surface of the portion of the body in forward flow appearing within the two envelopes of disturbance, Σ_v is the reference surface of any vortex sheets appearing within the two envelopes of disturbance, and $\boldsymbol{\nu}$ is the outward normal to Σ_b or Σ_v. The reference surface of the reverse body must lie on the same cylinder, if considered infinite, as Σ_b. The same is true for the two vortex sheets. For definiteness, we taken the reference surface of the reverse body to be identical with Σ_b *inside* the two envelopes of disturbance [see figure 3 (*b*) for an illustration]; but items (ii) and (iii) under (6) can ultimately be interpreted to represent any boundaries interior to the regions T. The reverse and forward bodies themselves may still be different.

It is easy to see that $\mathbf{V}_F \cdot \boldsymbol{\nu}$ is always evaluated on Σ_b by the boundary condition and that $\mathbf{U}_F \cdot \mathbf{V}_F$ on Σ_b is always the essential term in the pressure relation. Further the jumps in the relevant components of $\mathbf{V}_F$ across Σ_v can be evaluated. Therefore (20) evaluates, for all quasi-cylinders, the integral

$$\int_{\Sigma_b} \mathbf{U}_F \cdot \mathbf{V}_F \mathfrak{B}_R \cdot \boldsymbol{\nu} \, ds$$

in terms of the first-order forward and reverse flow. By suitable choice of $\mathfrak{B}_R \cdot \boldsymbol{\nu}$, we can evaluate second-order aerodynamic forces without solving the second-order problem, or we can formulate an integral equation to invert for $\mathbf{U}_F \cdot \mathbf{V}_F$ on Σ_b; in exchange we have to obtain the chosen reverse-flow and work out the indicated integrals.

It is now necessary to discuss these quantities in terms of an intrinsic co-ordinate system attached to the reference surface $\Sigma \equiv \Sigma_b + \Sigma_v$ instead of to the true surface $A \equiv A_b + A_v$. Since the generators of Σ are parallel to the x-axis, let $\alpha = \alpha(y,z)$ and $\beta = \beta(y,z)$ be orthogonal curvilinear co-ordinates. Let $\beta = 0$ define Σ, and let β increase in the direction of $\boldsymbol{\nu}$ [see figure 3 (*a*)]. The lines of constant α are perpendicular to Σ. Take the metric of the β-co-ordinate as unity so that β is also the distance from Σ; further, $\beta \gtreqless 0$. Let the metric of the α-co-ordinate be h. We then express the equation of A as

$$\beta = G(x, \alpha), \tag{21}$$

where G is of order τ. For the examples of zero-thickness wings and vortex sheets, A and Σ are both two-sided and we may distinguish when necessary between the sides with plus and minus signs as in the figure. The flow on A^+ is conventionally continued to Σ^+ analytically, and the flow on A^- is likewise connected analytically to the flow on Σ^-. The process transfers the jump surface from A to Σ, but the original physically significant one can be recovered later; however, the process fails at points like the tip, where the singularity resembles the one at sharp leading edges of wings. When Σ is two-sided, the (α, β) system covers the region twice, but it is easy to see which sheet to use. For example, (α^-, β^-) is identified with the flow on Σ^-, on which β^- increases along $\boldsymbol{\nu}^-$.

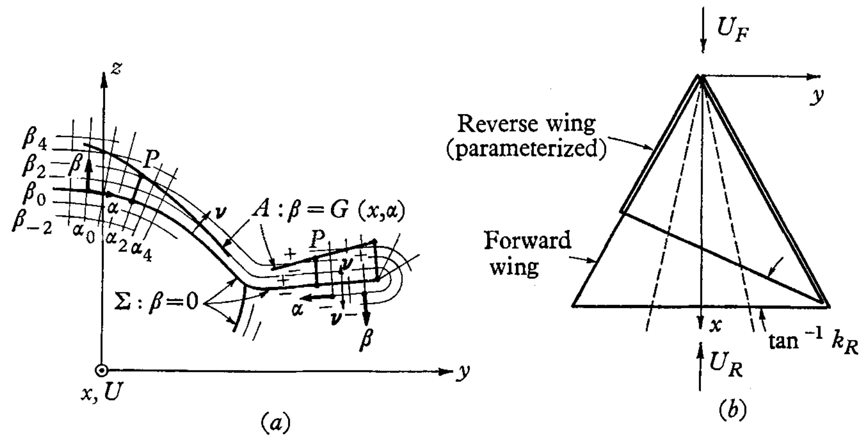

FIGURE 3. (*a*) Quasi-cylindrical surface A described by intrinsic co-ordinates attached to reference cylinder Σ: a section by x = const. (*b*) Determination of pressure distribution on delta wing for forward flow.

After expansion about the reference surface and replacement of ϕ by φ whenever the error statement permits, the tangency condition on the surface $\beta = G(x,\alpha)$ of a body or a vortex sheet leads us to

$$\phi_\beta = G_x(U+\varphi_x) + (1/h^2)\, G_\alpha \varphi_\alpha - G\varphi_{\beta\beta} \quad \text{on} \quad \Sigma \tag{22}$$

according to second-order theory, and, correspondingly, to

$$\varphi_\beta = UG_x \quad \text{on} \quad \Sigma \tag{23}$$

according to first-order theory (for either flow direction).

From (20) and (22), we have

$$\mathbf{V}_F . \boldsymbol{\nu} = \phi_{F\beta}(x,\alpha,0) \quad \text{on} \quad \Sigma_b, \tag{24}$$

and the component is evaluated in terms of the known forward-body geometry and first-order forward flow.

Using (23) and geometric considerations, we can express the term $\mathfrak{V}_R . \boldsymbol{\nu}$ in (20) as

$$\mathfrak{V}_R . \boldsymbol{\nu} = -\mathbf{U}_R \mathbf{n}_R . \mathbf{i}[1+O(\tau^2)] \quad \text{on} \quad \Sigma_b, \tag{25}$$

where $\mathbf{n}_R$ is the outward normal to the reverse-body surface $\beta = G_{bR}(x,\alpha)$. This expression is useful for evaluating the second-order drag; we would then choose $\mathbf{n}_R = \mathbf{n}_F$.

From the usual isentropic relation, the pressure on A may be written

$$[p-p_0]/\rho_0 = -\mathbf{U}.\mathbf{V}-\tfrac{1}{2}\mathfrak{V}.\mathfrak{W}, \tag{26}$$

to second order. Expansion about the reference surface gives

$$[p(A)-p_0]/\rho_0 = -U\phi_x - UG\varphi_{x\beta}-\tfrac{1}{2}\mathfrak{V}.\mathfrak{W} \quad \text{on} \quad \Sigma \tag{27}$$

for $U \gtrless 0$. To first order, this is merely

$$[p(A)-p_0]/\rho_0 = -U\varphi_x. \tag{28}$$

Since
$$\mathbf{U}_F.\mathbf{V}_F = U_F\phi_{Fx} \quad \text{on} \quad \Sigma_b, \tag{29}$$

in particular, we see from (27) that the only second-order term required to find p_F on A_b (and hence the aerodynamic forces) to second order is $\mathbf{U}_F.\mathbf{V}_F$ on Σ_b. And this is the first term in (20).

Before discussing the jumps across Σ_v, we first consider for illustration a simple geometry where no vortex sheets are present within the two envelopes of disturbance. Shown in figure 3(*b*) is a symmetric flat-plate delta wing with supersonic edges. The reference surface is the plane $z = 0$; $\beta \equiv z$ and $\alpha \equiv y$ for $z \geqslant 0$. If $(-\tau)$ is the incidence, then $G = (\tan\tau)x \approx \tau x$. The flow past each side is independent of the other, so that (20) may be applied to the side $z \geqslant 0$ only and Σ_b then refers only to this side of the reference surface. To solve for the surface pressure in forward flow, the reverse wing is chosen as shown. Because its supersonic leading edge has a variable sweep angle, viz. $\tan^{-1}k_R$, equation (20) is actually a family of relations and, in fact, formulates to a known integral equation in the current variable k_R. The components of $\mathfrak{V}_R$ are constant. Since the first-order forward flow is well known, $\mathbf{V}_F.\boldsymbol{\nu}$ is evaluated by (24) and (22) and the right member of (20) is evaluated through the definition of Q for quasi-cylinders. The inversion would give the conical variable $\mathbf{U}_F.\mathbf{V}_F$ on $z = 0$. This problem is considered in detail and solved analytically in the companion paper; the solution is then adjusted to make it uniform.

Across a vortex sheet
$$p(A_v^+)-p(A_v^-) = 0. \tag{30}$$

Using (27), we get

$$U(\phi_x^+-\phi_x^-) = -UG_v^+(\varphi_{x\beta+}^+ + \varphi_{x\beta-}^-)-\tfrac{1}{2}[\mathfrak{V}^+.\mathfrak{W}^+-\mathfrak{V}^-.\mathfrak{W}^-] \quad \text{on} \quad \sigma_v, \tag{31}$$

since $G_v^- = -G_v^+$. One side of Σ_v will be denoted by σ_v. We next form the jump $(\mathbf{V}^+-\mathbf{V}^-).\boldsymbol{\nu}^+$ on σ_v by applying (22) to both sides and adding

$$\phi_{\beta+}^+ + \phi_{\beta-}^- = G_{vx}^+(\varphi_x^+-\varphi_x^-)+\frac{1}{(h^+)^2}G_{v\alpha+}^+\varphi_{\alpha+}^+ + \frac{1}{(h^-)^2}G_{v\alpha-}^-\varphi_{\alpha-}^-$$
$$-G_v^+(\varphi_{\beta+\beta+}^+ - \varphi_{\beta-\beta-}^-), \quad \text{on} \quad \sigma_v. \tag{32}$$

Since $G_{vx}^- = -G_{vx}^+$, the first term on the right of (22) [the only one which is $O(\tau)$] cancels. In the right members of (31) and (32), it is therefore permissible, to second order, to use the first-order results for $G_v(x,\alpha)$. The two jumps are consequently evaluated in terms of first-order results! The corresponding first-order jump conditions in forward or reverse flow are

$$\varphi_x^+ - \varphi_x^- = 0 \quad \text{on} \quad \sigma_v, \tag{33}$$

from (28) and (30), and

$$\varphi^+_{\beta+} + \varphi^-_{\beta-} = 0 \quad \text{on} \quad \sigma_v, \tag{34}$$

from (23). To show how these results enter, we write the integral over Σ_v in (20) in the jump form and use (33) and (34) for the reverse flow. The result is

$$\int_{\sigma_v} [(\mathbf{U}_F.\mathbf{V}^+_F - \mathbf{U}_F.\mathbf{V}^-_F)\,\mathfrak{V}_R.\mathbf{v}^+ - \mathbf{U}_R.\mathfrak{V}_R(\mathbf{V}^+_F - \mathbf{V}^-_F).\nu^+]\,ds. \tag{35}$$

The two jumps in (35) are evaluated by (31) and (32), respectively, when these are interpreted to apply to the forward flow. If a region of σ_v is occupied by vortex sheets of the reverse flow but not of the forward flow, the integrand vanishes thereon because the left numbers of (31) and (32) must vanish. A contribution to (35) occurs when vortex sheets of the forward flow or both flows are present in a region. We may therefore reinterpret Σ_v in (20) to signify the reference surface of any vortex sheets of the *forward flow* appearing within the two envelopes of disturbance. The theory does not account for vortex instability and roll-up. This neglect is acceptable near the trailing edge, and only the near sheets enter the reverse-flow relation.

The right members of (31) and (32) are simplified when (33) and (34) are inserted therein. Further, we wish to carry out the details in terms of the second-order *incremental* improvement to first-order results and therefore introduce the second-order potential f defined by

$$\phi = \varphi + f, \quad f = O(\tau^2). \tag{36}$$

Then our final results for vortex sheets in forward flow are

$$f^+_{Fx} - f^-_{Fx} = -G^+_{vF}(\varphi^+_{Fx\beta+} + \varphi^-_{Fx\beta-}) - \frac{1}{2U_F}\left[\frac{1}{(h^+)^2}(\varphi^+_{F\alpha+})^2 - \frac{1}{(h^-)^2}(\varphi^-_{F\alpha-})^2\right] \quad \text{on} \quad \sigma_v \tag{37}$$

from the continuity of the pressure, and

$$f^+_{F\beta+} + f^-_{F\beta-} = \frac{1}{(h^+)^2}G^+_{vF\alpha+}\varphi^+_{F\alpha+} + \frac{1}{(h^-)^2}G^-_{vF\alpha-}\varphi^-_{F\alpha-} - G^+_{vF}(\varphi^+_{F\beta+\beta+} - \varphi^-_{F\beta-\beta-}) \quad \text{on} \quad \sigma_v \tag{38}$$

from the tangency of the flow.

5. Second-order drag of an arbitrary quasi-cylinder

Using the geometric framework and discussion of the last section, we can write down at once the second-order drag of an arbitrary quasi-cylinder. From (19), (27) and (36), we obtain the defining expression for the drag in forward flow, D_F,

$$D_F = \mathscr{D}_F + \rho_0 U_F \int_{\Sigma_b} f_{Fx}\,\mathbf{n}.\mathbf{i}\,ds - \rho_0 U_F \int_{\Sigma_b} \varphi_{Fx\beta} G_b G_{bx}\,ds - \tfrac{1}{2}\rho_0 \int_{\Sigma_b} \mathfrak{V}_F.\mathfrak{W}_F G_{bx}\,ds. \tag{39}$$

The first-order drag in forward flow is denoted by $\mathscr{D}_F$. Of the three integrals giving the second-order increment, the last two are seen to depend on first-order results. Noting the first integral in (20), (25) and (36), we see we want

$$\mathbf{n}_R \equiv \mathbf{n}_F \equiv \mathbf{n}$$

and choose the body in reverse flow to be the same as the one of interest in forward flow. We write (20) in terms of potentials, subtract from it its first-order counterpart, utilize the discussion of the last section, and find immediately the reverse-flow result

$$\rho_0 U_F \int_{\Sigma_b} f_{Fx} \mathbf{n}.\mathbf{i}\, ds = -\rho_0 \int_{\Sigma_b} \varphi_{Rx}\left(G_{bx}\varphi_{Fx} + \frac{1}{h^2} G_{b\alpha}\varphi_{F\alpha} - G_b \varphi_{F\beta\beta}\right) ds$$
$$-\rho_0 \int_{\sigma_v} [(f^+_{Fx} - f^-_{Fx})\, \varphi^+_{R\beta+} + \varphi^+_{Rx}(f^+_{F\beta+} + f^-_{F\beta-})]\, ds$$
$$-\rho_0 \int_T \varphi_{Rx} Q\, dT, \tag{40}$$

where the two jumps across σ_v are evaluated by (37) and (38). Thus the single integral in (39) depends on the second-order solution is expressed by (40) in terms of the geometry and the first-order forward and reverse flows past the body of interest.

An elementary calculation verifies that (39) and (40) lead to the same formula for the drag of an arbitrary two-dimensional section that one obtains from the Busemann second-order theory (see e.g. Lighthill 1954).

6. Second-order transverse forces and pitching moments on an arbitrary quasi-planar wing

Consider a wing whose reference surface consists of the two sides of the plane $z = 0$ within the projection thereon of the planform. Suppose the upper and lower surfaces are denoted simply by

$$z = {}^+Z(x,y) \quad \text{and} \quad {}^-Z(x,y), \tag{41}$$

respectively. We wish to determine the transverse force supported by some part or all of its planform in forward flow, i.e. the partial or entire lift. We also want the pitching moment contributed by this part of the planform or the entire planform. Consider a cylinder with generators parallel to z that cuts out a portion A^* of A_b and a portion σ^* of σ_b, where σ_b is one side of the wing reference surface; A^* might denote the surfaces of a flap or, of course, the entire wing. Using (27) and (36), we obtain the defining expression for the second-order transverse force L^*_F contributed by A^* in forward flow

$$L^*_F = -\int_{A^*} (p_F - p_0)\, \mathbf{n}_F . \mathbf{k}\, ds$$
$$= \mathscr{L}^*_F + \rho_0 U_F \int_{\sigma^*} (f^+_{Fx} - f^-_{Fx})\, ds + \rho_0 U_F \int_{\sigma^*} (\varphi^+_{Fxz}{}^+Z - \varphi^-_{Fxz}{}^-Z)\, ds$$
$$+ \tfrac{1}{2}\rho_0 \int_{\sigma^*} (\mathfrak{V}^+_F . \mathfrak{W}^+_F - \mathfrak{V}^-_F . \mathfrak{W}^-_F)\, ds. \tag{42}$$

Here $\mathscr{L}^*_F$ is the first-order transverse force contributed by A^* in forward flow. Let $\mathbf{r}$ be the position vector. Then the defining expression for the second-order pitching moment contributed by A^* in forward flow, viz.

$$M^*_F = -\mathbf{j} . \int_{A^*} \mathbf{r} \times (p_F - p_0)\, \mathbf{n}_F\, ds,$$

is almost the same as (42): replace $L_F^* - \mathscr{L}_F^*$ by $M_F^* - \mathscr{M}_F^*$ and multiply the three integrands by $(-x)$. Here $\mathscr{M}_F^*$ is the first-order version of M_F^*. Since the remaining integrals follow from the first-order solution for the wing of interest, we require

$$\rho_0 U_F \int_{\sigma^*} (f_{Fx}^+ - f_{Fx}^-)\, ds \quad \text{and} \quad -\rho_0 U_F \int_{\sigma^*} (f_{Fx}^+ - f_{Fx}^-)\, x\, ds \tag{43}$$

to find L_F^* and M_F^*, respectively.

It is simple to contrive a reverse flow enabling the construction of these two integrals in (20); their evaluation then follows from the discussion of §4. Introduce the potentials in (20), subtract its first-order counterpart, and write the result in the jump form across $z = 0$. The result is

$$\int_{\sigma_b + \sigma_v} (f_{Fx}^+ \varphi_{Rz}^+ - f_{Fx}^- \varphi_{Rz}^- + \varphi_{Rx}^+ f_{Fz}^+ - \varphi_{Rx}^- f_{Fz}^-)\, ds = -\int_T \varphi_{Rx} Q\, dT. \tag{44}$$

After comparing the first two terms on the left (on σ_b) with (43), we choose a lifting wing with the same planform but without thickness for the reverse body, because the boundary condition then is

$$\varphi_{Rz}^+ = \varphi_{Rz}^- = U_R Z_{bRx} \equiv -U_R \alpha_R(x,y) \quad \text{on} \quad \sigma_b. \tag{45}$$

Here we have also introduced the angle of attack α_R. Since this is still arbitrary, we can construct the integrals in (43) as well as others like them. Because (45) implies a solution for the potential φ_R that is odd in z, we obtain the property

$$\varphi_{Rx}^- = -\varphi_{Rx}^+ \quad \text{on} \quad \sigma_b, \tag{46}$$

while equations (33) and (34) become

$$\varphi_{Rx}^- = \varphi_{Rx}^+ = 0 \quad \text{and} \quad \varphi_{Rz}^- = \varphi_{Rz}^+ \quad \text{on} \quad \sigma_v. \tag{47}$$

The first half of (47) suppresses in (44) one of the vortex discontinuities in forward flow. In accord with the discussion in §4, we use (22), (23), (36) and (37) in (44) and obtain the desired reverse-flow result

$$\rho_0 U_F \int_{\sigma_b} (f_{Fx}^+ - f_{Fx}^-)\, \alpha_R\, ds = -\rho_0 \int_{\sigma_b} \varphi_{Rx}^+ (f_{Fz}^+ + f_{Fz}^-)\, ds - \rho_0 \int_{\sigma_v} (f_{Fx}^+ - f_{Fx}^-)\, \varphi_{Rz}\, ds$$
$$-\rho_0 \int_T \varphi_{Rx} Q\, dT, \tag{48a}$$

where

$$f_{Fz}^+ + f_{Fz}^- = {}^+Z_x \varphi_{Fx}^+ + {}^-Z_x \varphi_{Fx}^- + {}^+Z_y \varphi_{Fy}^+ + {}^-Z_y \varphi_{Fy}^- - {}^+Z \varphi_{Fzz}^+ - {}^-Z \varphi_{Fzz}^- \quad \text{on} \quad \sigma_b \tag{48b}$$

and

$$f_{Fx}^+ - f_{Fx}^- = -Z_{vF}(\varphi_{Fxz}^+ - \varphi_{Fxz}^-) - \frac{1}{2U_F}(\varphi_{Fy}^{+2} - \varphi_{Fy}^{-2}) \quad \text{on} \quad \sigma_v. \tag{48c}$$

The terms on the right side of (48a) depend on only the geometry and the two first-order flows. Of course, the integral over σ_v might well be absent for the particular wing of interest.

Now to construct in (48a) the first integral in (43), let

$$\alpha_R = \begin{Bmatrix} k & \text{on} & \sigma^* \\ 0 & \text{on} & \sigma_b - \sigma^* \end{Bmatrix}, \tag{49a}$$

where k is a constant which then will cancel out because $\varphi_R \sim k$. To construct the second integral in (43), let

$$\alpha_R = \begin{Bmatrix} -kx & \text{on} & \sigma^* \\ 0 & \text{on} & \sigma_b - \sigma^* \end{Bmatrix}. \tag{49b}$$

This completes the demonstration. Evaluation of the three integrals on the right side of (48a) is facilitated by the usual decomposition of φ_F into the thickness and lift problems

$$\varphi_F = \varphi_{F1} + \varphi_{F2}.$$

The first is an even function of z and the second is an odd function so that simplifications due to symmetry and antisymmetry appear; T is symmetric about $z = 0$; the form of Q is simplified. Some further techniques for evaluation of these integrals are presented in the companion paper.

In §§5 and 6 we have evaluated (in terms of the first-order flows) certain second-order aerodynamic forces on quasi-cylinders using (20). These are presented as illustrations. Clearly, other results of similar nature may be deduced as required.

7. Adjustment of expressions for second-order forces

These expressions for the second-order forces and moments on quasi-cylinders, obtained by use of the reverse-flow results with the defining expressions, must be examined and adjusted for defects in the first-order theory before they are in accord with a uniform (or more uniform) second-order theory. This has to be done for the specific problem at hand. It would be sufficient to make the first-order result uniform throughout the region but this is not generally necessary in order to make just the *integrals* correct. One well-known illustration of this point is the fact that the edge forces on wings may be determined correctly to first-order from the ordinary non-uniform solution by enclosing the singular edge with a vanishing tubular surface (see Ward 1955).

The nature of the procedure required is better appreciated by considering the integral over the body of the pressure as it would be given, in turn, by ordinary and uniform first-order theory, and by ordinary and uniform second-order theory. Whereas there is no one-to-one correspondence of integration regions in this direct calculation and the indirect reverse-flow calculation of second-order force, it is clear that the nature and treatment of the singularities must be the same in each. Thus, the expressions we have deduced above might not even converge without adjustment, much less yield the correct result. Considering the class of problems wherein the singularities are associated with the wave phenomena (perhaps the most interesting class), two specific comparisons among such pressure distributions are provided by Wallace & Clarke (1963) and Clarke & Wallace (1963). In this class of problems, it appears that the adjustments to ordinary second-order theory make, in general, only higher-order contributions to the forces and moments and may be disregarded. This implies that the expressions deduced for the second-order forces may be evaluated as they stand using ordinary first-order results, although it might be necessary to remove the singularities in these first-order results, e.g. by expanding about singular points and deleting singular terms or by smoothing. But local adjustment of the first-order wave system is generally necessary when computing directly the corresponding first-order forces; *the contributions that render the solution uniform* add terms to the forces that can be of the same order as the second-order terms discussed. The use of Whitham's Shift Rule for this purpose can readily be accomplished graphically. While it is true that the first-order adjustments can make second-order contributions to the force throughout the region, if included, the contributions away from the singular surfaces are spurious; this is proved by Wallace & Clarke (1963, see equation (43)).

This work has been sponsored by the U.S. Bureau of Naval Weapons, through contract NOw 62-0604-c under Brown University Sub-contract No. 77797 and has been coordinated within the Bumblebee Project through the Applied Physics Laboratory, The Johns Hopkins University.

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